MATH 2010B Tutorial 2. 21/09

Outline :

- 2 descriptions of shapes in $\mathbb{R}^n$

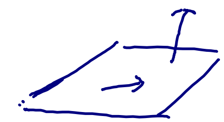
  (focus on flat spaces. i.e. linear obj)

  for example : line in $\mathbb{R}^3$, plane in $\mathbb{R}^3$.

- Calculating distance between two linear obj in $\mathbb{R}^n$.

- Q: ① how to define a $\overset{\overset{S}{\triangleq}}{\text{plane}}$ in $\mathbb{R}^3$?

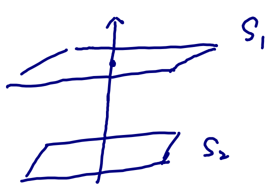
  ② how to define a line in $\mathbb{R}^3$?



$\subset \mathbb{R}^3$

answer 1: one normal vector & a point.

$ax+by+cz = \text{constant}$

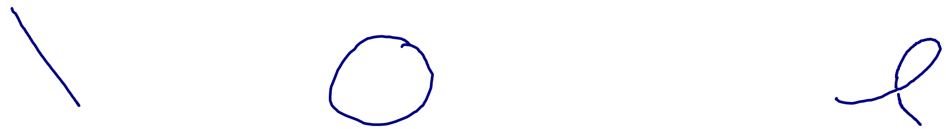


answer 2: 2 vector $\vec{v}_1, \vec{v}_2$ in S & a point.

not parallel.

Central problem of interest

To describe "good" shapes in $\mathbb{R}^n$ mathematically



2 approaches :

$S \subset \mathbb{R}^n$

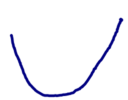
① Parametrization

$\mathbb{R}^m \xrightarrow{r} \mathbb{R}^n$

· S as image of r

i.e $S = \{ r(\vec{t}) : \vec{t} \in \mathbb{R}^m \}$

· e.g. parabola



$\mathbb{R} \xrightarrow{r} \mathbb{R}^2$

$x \longmapsto r(x) = (x, ax^2+bx+c)$

· telling allowed directions to move within S.

② Equations

$\xrightarrow{f} \mathbb{R}^k$

· S as preimage of f at some $\vec{y}$ in $\mathbb{R}^k$

i.e $S = \{ \vec{x} \in \mathbb{R}^n : f(\vec{x}) = \vec{y} \}$

e.g. $S \subset \mathbb{R}^3$

$ax+by+cz = \text{constant}$

· telling prohibited directions to move within S.

✱ In the aforementioned approaches.

Type of funct r, f that are used $\xrightarrow{\text{determine}}$ kind of geo shape in $\mathbb{R}^n$ that can be described.

- Smooth funct (differentiable ∞ many times) $\longrightarrow$ smooth shapes (Mathematicians call them manifolds. not to be discussed here)

- Affine transformations $\longrightarrow$ flat shapes i.e linear obj e.g lines. planes

  $f(x) = A\vec{x} + \vec{b}$

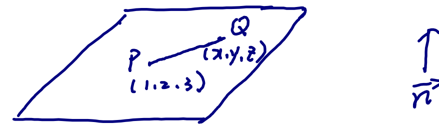
  A : matrix

  $\vec{b}$: vector

e.g $\underline{ax + by + cz = \text{constant}}$ $\longrightarrow$ planes $\subset \mathbb{R}^3$

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

e.g. S: plane $\subseteq \mathbb{R}^3$



$\vec{n} = (a,b,c)$

① $\underline{ax+by+cz = \text{constant}}$

$\vec{n} = (2,3,5)$ & $p = (1,2,3) \in S$

$(x,y,z) \in \mathbb{R}^3$

if $\underset{\underset{Q}{\parallel}}{(x,y,z)} \in S$,

$\overrightarrow{PQ} \perp \vec{n} \iff \overrightarrow{PQ} \cdot \vec{n} = 0$

$\parallel$

$(x-1,\ y-2,\ z-3)$

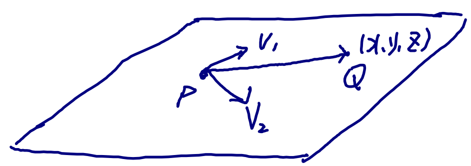
f: $\underline{(x-1)\cdot 2 + (y-2)\cdot 3 + (z-3)\cdot 5 = 0}$

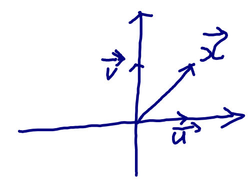
In general $(x_0, y_0, z_0) \in S$, $\vec{n} = (a,b,c)$

$$(x-x_0)\cdot a + (y-y_0)\cdot b + (z-z_0)\cdot c = 0$$

$$\overrightarrow{PQ} \cdot \vec{n} = 0$$

② parametrization ?





$\underline{\vec{x} = a\vec{u} + b\vec{v}}$

$$\overrightarrow{PQ} = \underset{s \in \mathbb{R}}{s\vec{v_1}} + \underset{t \in \mathbb{R}}{t\vec{v_2}}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = s \underbrace{\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}}_{= \vec{v_1}} + t \underbrace{\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}}_{= \vec{v_2}}$$

One remark.:

we can find: $\vec{v_1}'$, $\vec{v_2}'$, p'

s.t $\overrightarrow{p'Q} = s\vec{v_1}' + t\vec{v_2}'$

Distance between hyperplanes

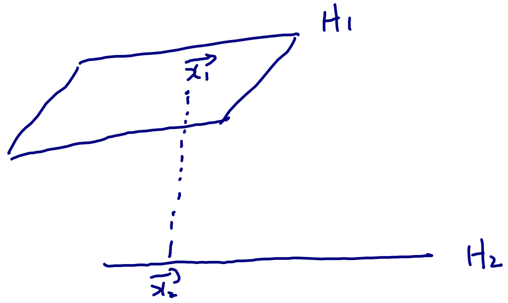
✡ Fact :

Given two hyperplane H_1, H_2 in $\mathbb{R}^n$,

$\exists$ points $\vec{x}_1 \in H_1$ & $\vec{x}_2 \in H_2$. s.t

$\forall$ $\vec{y}_1 \in H_1$ &, $\vec{y}_2 \in H_2$

$$\|\vec{x}_1 - \vec{x}_2\| \le \|\vec{y}_1 - \vec{y}_2\|$$



we call $\|\vec{x}_1 - \vec{x}_2\|$ the distance between H_1, H_2.

change : show that $\vec{x}_1 - \vec{x}_2 \perp H_1$ & $\vec{x}_1 - \vec{x}_2 \perp H_2$

e.g. 1 Find the distance between

$A = (2, 1, 1)$ & the plane $P: -x+2y-z=-4$

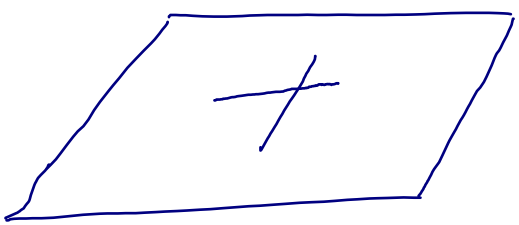
e.g. 2 : Find the distance between lines

$L_1(s) = (-4, 9, -4) + s(4, -3, 0)$

$L_2(t) = (5, 2, 10) + t(4, 3, 2)$

r.k. distance between two line can be 0.

e.g



Sol of e.g. 2

1. Find A on L_1, B on L_2, s.t

$$\overrightarrow{AB} \perp L_1, L_2$$

$A, B,$ is of the form

$$A = \begin{pmatrix} -4 \\ 9 \\ -4 \end{pmatrix} + s\begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 5 \\ 2 \\ 10 \end{pmatrix} + t\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}, \quad s, t \in \mathbb{R}$$

Solve

$$\begin{pmatrix} 4 & -3 & 0 \\ 4 & 3 & 2 \end{pmatrix} \Bigg[\underbrace{\begin{pmatrix} -4 \\ 9 \\ -4 \end{pmatrix} + s\begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ 10 \end{pmatrix} - t\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}}_{\overrightarrow{AB}} \Bigg] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

tangent vector to L_1, L_2 resp

Re arranging.

$$\begin{pmatrix} 4 & -3 & 0 \\ 4 & 3 & 2 \end{pmatrix} \begin{pmatrix} 4 & -4 \\ -3 & -3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 4 & -3 & 0 \\ 4 & 3 & 2 \end{pmatrix} \begin{pmatrix} 9 \\ -7 \\ 14 \end{pmatrix}$$

on solving $(s, t) = (2, -1)$

2° Find $\|\overrightarrow{AB}\|$ (easy. do it yourself)

Answer : The distance $= \|\overrightarrow{AB}\| = 13.$